

# Vanishing order of solutions to Schrödinger equation

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## Abstract

On a compact manifold, we give a Carleman estimate on the operator  $\Delta + W$  when  $\Delta$  is the laplacian operator and  $W$  is a bounded function. We then deduce quantitative uniqueness result for solutions to

$$\Delta u + Wu = 0$$

using doubling estimates. In particular this show that the vanishing order is everywhere less than  $C_1 \|W\|_\infty^{\frac{2}{3}} + C_3$ . Finally we investigate the sharpness of this results by constructing a potential  $W$  with compact support on  $\mathbb{S}^2$ .

## 1 Introduction and statement of the results

Let  $(M, g)$  be a compact, connected,  $n$ -dimensional smooth Riemannian manifold,  $\Delta$  the Laplace operator on  $M$  and  $W$  a bounded function on  $M$ . If  $u$  is a non trivial solution to

$$\Delta u + Wu = 0, \tag{1.1}$$

we consider the possible vanishing order, depending on  $W$ , of  $u$  in any point. In the case that  $W$  is a constant, *i.e.*, when dealing with the eigenfunctions of the Laplacian, it is a well known result of H. Donnelly and C. Fefferman [3] that the vanishing order is everywhere bound by  $c\sqrt{\lambda}$ , with  $c$  constant depending only on  $M$ . When  $W$  is a  $C^1$  function, the author have shown in [1] that the vanishing order is bounded by  $C_1 \sqrt{\|W\|_{C^1}} + C_2$ , where  $\|W\|_{C^1} = \sup_M |W| + \sup_M |\nabla W|$  and the norm  $|\nabla W|$  is taking with respect to the metric  $g$ . In this paper we investigate the possible vanishing order of solutions to (1.1) when  $W$  is only a bounded function. For  $W$  a real valued function, I. Kukavica established, in [13], the following uniform upper bound :

$$C(\sup(W_-)^{\frac{1}{2}} + (\text{osc}(W))^2 + 1)$$

where  $W_- = \min(W, 0)$  and  $\text{osc}(W) = \sup W - \inf W$ . In particular the vanishing order of solutions to (1.1) is uniformly bounded by

$$C(1 + \|W\|_\infty^2).$$

Our first goal is to sharpen this last estimate. We shall prove the following:

**Theorem 1.1.** *There exist two non-negatives constants  $C_1, C_2$  depending only on  $M$ , such that, for any solutions  $u$  to (1.1) and for any point  $x_0$  in  $M$ , the vanishing order of any non-zero solutions to (1.1) is everywhere bounded by*

$$C_1 \|W\|_{\infty}^{\frac{2}{3}} + C_2.$$

Note here that the potential  $W$  and the solution  $u$  may take their values in  $\mathbb{C}$ . This result is sharp in the following sense, considering complex valued solutions of (1.1), the exponent  $2/3$  on  $\|W\|_{\infty}$  is the lowest one can obtain in the upper bound on the vanishing order of solutions. More precisely we will show the following :

**Theorem 1.2.** *There exists a constant  $C$  such that, if  $N > 0$  is an arbitrary great number, there exists a function  $W \in L^{\infty}(\mathbb{S}^2, \mathbb{C})$  with  $N \geq C \|W\|_{\infty}^{\frac{2}{3}}$  and  $u \in \mathcal{C}^2(\mathbb{S}^2, \mathbb{C})$  solution to (1.1) which vanishes with order  $N$  in  $P$ . Moreover  $W$  can be choosen of compact support with*

$$\text{supp}(W) \subset (\mathbb{S}^2 \setminus (\{P\} \cup \{Q\}))$$

where the points  $P, Q$  are antipodals.

In the case one only consider real valued functions, the upper bound

$$C_1 \|W\|_{\infty}^{\frac{1}{2}} + C_2$$

is expected (see by example [13]). However this seems to be a difficult problem which, to the knowledge of the author, has not be solved yet. As mentionned in [2], since Carleman estimates don't distinguish between the real and complex case, it seems difficult to obtain such result with this method.

The method of [13] was based on the frequency function [4, 14] when our's relies on Carleman estimates [3, 8, 10, 11, 12, 16, 17, ...]. This two methods are the principal way to obtain quatitative uniqueness results for solutions of partial differential equations. The first section is devoted to study the vanishing order of solutions to (1.1), by using Carleman inequalities. This is inspired by the works of H. Donnelly and C. Fefferman for eigenfunctions of the Laplacian. In particular in section 2 we first established a Carleman estimate which is only true for great enough paramater  $\tau \geq \tau_0$  and we state explicitly how  $\tau_0$  depends on the potential  $W$ . We proove this estimate by elementary method and for a certain family of weight functions. Then we used the Carleman estimate with a special choice of weight functions to obtain a Hadamard's type three circles theorem and doubling inequalities on solution of (1.1). This will proove theorem 1.1. Section 3 is devoted to proove theorem 1.2. We construct a sequence  $(u_k, W_k)$  verifying  $\Delta u_k + W_k u_k = 0$  on the two dimensionnal sphere which shows that the sharp upper bound  $C_1 \|W\|_{\infty}^{\frac{2}{3}} + C_2$  on the vanishing order can be obtained with  $W_k$  of support in the neighborhood of two antipodals points.

It must be emphasis that both the upper bound and his sharpness are already known. In [10], C. A. Kenig established this results, with similar works on  $\mathbb{R}^n$  which clearly imply the present results of this paper.

### Notations

For a fixed point  $x_0$  in  $M$  we will use the following standard notations:

- $r := r(x) = d(x, x_0)$  the Riemannian distance from  $x_0$ ,
- $v_g$  the volume form induced by  $g$ ,
- $B_r := B_r(x_0)$  the geodesic ball centered at  $x_0$  of radius  $r$ ,
- $A_{r_1, r_2} := B_{r_2} \setminus B_{r_1}$ ,
- $\|\cdot\|$  the  $L^2$  norm on  $M$  and  $\|\cdot\|_A$  the  $L^2$  norm on the (measurable) set  $A$ . In case  $T$  is a vector field (or a tensor), it has to be understood as  $\|T\|_g$ .
- $c, C, c_i$  and  $C_i$  for  $i = 1, 2, \dots$  are constants which may depend on  $(M, g)$  and other quantities such that the weight functions in Carleman estimates (section 2.1), but not on the potential  $W$  or solutions  $u$ . Their values might change from one line to another.

## 2 Vanishing order

Recall that Carleman estimates are weighted integral inequalities with a weight function  $e^{\tau\phi}$ , where the function  $\phi$  satisfy some convexity properties, see by example [6, 7, 9]. We first prove in section 2.1 an  $L^2$ , singular weighted, Carleman inequality on the operator  $\Delta + W$ , for some class of weight functions satisfying convenient properties (see (2.1) and (2.2) below). After that, in sections 2.2 and 2.3, we make a particular choice of weight function which will allow us to derive a doubling estimate on solutions to (1.1).

### 2.1 Carleman estimate

Let us first define the class of (singular) weight functions we will work with. Let  $f : ]-\infty, T[ \rightarrow \mathbb{R}$  of class  $\mathcal{C}^3$ , and assume there are constants  $\mu_i > 0$ ,  $i = 1, \dots, 4$ , such that :

$$\begin{aligned} 0 < \mu_1 \leq f'(t) \leq \mu_2 \\ \mu_3 |f^{(3)}(t)| \leq -f''(t) \leq \mu_4, \quad \forall t \in ]-\infty, T[ \\ \lim_{t \rightarrow -\infty} -e^{-t} f''(t) = +\infty \end{aligned} \quad (2.1)$$

**Example :** It is clear that the functions defined by  $f_\varepsilon(t) = t - e^{\varepsilon t}$ , satisfy the conditions (2.1) provided  $0 < \varepsilon < 1$  and  $T$  is a large negative number.

Finally we define our weight function as

$$\phi(x) = -f(\ln r(x)). \quad (2.2)$$

Now we can state the main result of this section:

**Theorem 2.1.** *There exist positive constants  $R_0, C, C_1, C_2$ , which depend only on  $M$  and  $f$ , such that, for any  $x_0 \in M$ , any  $\delta \in (0, R_0)$ , any  $W \in L^\infty(M)$ , any  $u \in C_0^\infty(B_{R_0}(x_0) \setminus B_\delta(x_0))$  and any  $\tau \geq C_1 \|W\|_\infty^{\frac{2}{3}} + C_2$ , one has*

$$\begin{aligned} C \left\| r^2 e^{\tau\phi} (\Delta + W) u r^{-n/2} \right\|^2 &\geq \tau^3 \left\| \sqrt{|f''(\ln r)|} e^{\tau\phi} u r^{-n/2} \right\|^2 \\ &+ \tau^2 \delta \left\| r^{-\frac{1}{2}} e^{\tau\phi} u r^{-n/2} \right\|^2 + \tau \left\| r \sqrt{|f''(\ln r)|} e^{\tau\phi} |\nabla u|^2 r^{-n/2} \right\|^2. \end{aligned} \quad (2.3)$$

The following lemma contains the crucial part of theorem 2.1:

**Lemma 2.2.** *There exist positive constants  $R_0, C, C_1$ , which depend only on  $M$  and  $f$ , such that, for any  $x_0 \in M$ , any  $u \in C_0^\infty(B_{R_0}(x_0) \setminus \{x_0\})$  and any  $\tau \geq C_1$ , one has*

$$C \left\| r^2 e^{\tau\phi} |\Delta u| r^{-n/2} \right\|^2 \geq \tau^3 \left\| \sqrt{|f''(\ln r)|} e^{\tau\phi} u r^{-n/2} \right\|^2 + \tau \left\| r \sqrt{|f''(\ln r)|} e^{\tau\phi} |\nabla u| r^{-n/2} \right\|^2. \quad (2.4)$$

**Remark 2.3.** The exponent 3 on  $\tau$  in lemma 2.2 will play an important role in the following (compare to [3]). Like in [3] the important statement in theorem 2.1 is the one involving the parameter  $\tau$  :

$$\tau \geq C_1 \|W\|^{\frac{2}{3}} + C_2.$$

*proof of lemma 2.2.* Without loss of generality, we may suppose that all functions are real. We now introduce the polar geodesic coordinates  $(r, \theta)$  near  $x_0$ . Using Einstein notation, the Laplace operator takes the form :

$$r^2 \Delta u = r^2 \partial_r^2 u + r^2 \left( \partial_r \ln(\sqrt{\gamma}) + \frac{n-1}{r} \right) \partial_r u + \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u),$$

where  $\partial_i = \frac{\partial}{\partial \theta_i}$  and for each fixed  $r$ ,  $\gamma_{ij}(r, \theta)$  is a metric on  $\mathbb{S}^{n-1}$  and  $\gamma = \det(\gamma_{ij})$ .

Since  $(M, g)$  is smooth, we have for  $r$  small enough :

$$\begin{aligned} \partial_r(\gamma^{ij}) &\leq C(\gamma^{ij}) \quad (\text{in the sense of tensors}); \\ |\partial_r(\gamma)| &\leq C; \\ C^{-1} \leq \gamma &\leq C. \end{aligned} \quad (2.5)$$

Set  $r = e^t$ , the function  $u$  is then supported in  $] -\infty, T_0[ \times \mathbb{S}^{n-1}$ , where  $|T_0|$  will be chosen large enough. In this new variables, we can write :

$$e^{2t} \Delta u = \partial_t^2 u + (n-2 + \partial_t \ln \sqrt{\gamma}) \partial_t u + \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u).$$

The conditions (2.5) become

$$\begin{aligned} \partial_t(\gamma^{ij}) &\leq C e^t(\gamma^{ij}) \quad (\text{in the sense of tensors}); \\ |\partial_t(\gamma)| &\leq C e^t; \\ C^{-1} \leq \gamma &\leq C. \end{aligned} \quad (2.6)$$

It will be useful to note that this properties imply

$$|\partial_t \ln \sqrt{\gamma}| \leq C e^t \quad (2.7)$$

Now we introduce the conjugate operator :

$$\begin{aligned} L_\tau(u) &= e^{2t} e^{\tau\phi} \Delta(e^{-\tau\phi} u) \\ &= \partial_t^2 u + (2\tau f' + n-2 + \partial_t \ln \sqrt{\gamma}) \partial_t u \\ &\quad + \left( \tau^2 f'^2 + (n-2)\tau f' + \tau f'' + \tau \partial_t \ln \sqrt{\gamma} f' \right) u \\ &\quad + \Delta_\theta u, \end{aligned} \quad (2.8)$$

with

$$\Delta_\theta u = \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u).$$

It will be useful for us to introduce the following  $L^2$  norm on  $] - \infty, T_0[ \times \mathbb{S}^{n-1}$ :

$$\|V\|_f^2 = \int_{]-\infty, T_0[ \times \mathbb{S}^{n-1}} |V|^2 \sqrt{\gamma} f'^{-3} dt d\theta,$$

where  $d\theta$  is the usual measure on  $\mathbb{S}^{n-1}$ . The corresponding inner product is denoted by  $\langle \cdot, \cdot \rangle_f$ , *i.e*

$$\langle u, v \rangle_f = \int uv \sqrt{\gamma} f'^{-3} dt d\theta.$$

We will estimate from below  $\|L_\tau u\|_f^2$  by using elementary algebra and integrations by parts. We are concerned, in the computation, by the power of  $\tau$  and exponential decay when  $t$  goes to  $-\infty$ . First by triangular inequality one has

$$\|L_\tau(u)\|_f^2 \geq \frac{1}{2} I - II, \quad (2.9)$$

with

$$\begin{aligned} I &= \left\| \tau^2 f'^2 u + (n-2) \tau f' u + \partial_t^2 u + 2\tau f' \partial_t u + \Delta_\theta u \right\|_f^2, \\ II &= \left\| \tau f'' u + \partial_t \ln \sqrt{\gamma} f' u + (n-2) \partial_t u + \partial_t \ln \sqrt{\gamma} \partial_t u \right\|_f^2. \end{aligned} \quad (2.10)$$

We will now derive a lower bound of  $I$ . ( We will able to absorb  $II$  in  $I$  later.) To compute  $I$  we write :

$$I = I_1 + I_2 + I_3,$$

with

$$\begin{aligned} I_1 &= \left\| \partial_t^2 u + (\tau^2 f'^2 + (n-2) \tau f') u + \Delta_\theta u \right\|_f^2 \\ I_2 &= \|2\tau f' \partial_t u\|_f^2 \\ I_3 &= 2 \left\langle 2\tau f' \partial_t u, \partial_t^2 u + (\tau^2 f'^2 + (n-2) \tau f') u + \Delta_\theta u \right\rangle_f \end{aligned} \quad (2.11)$$

In order to make explicit the estimation of  $I_3$  we write it in a convenient way:

$$I_3 = J_1 + J_2 + J_3 \quad (2.12)$$

where, using  $2\partial_t u \partial_t^2 u = \partial_t (|\partial_t u|^2)$ , the integrals  $J_i$  are defined by :

$$J_1 = \int (2\tau f') \partial_t (|\partial_t u|^2) f'^{-3} \sqrt{\gamma} dt d\theta, \quad (2.13)$$

$$J_2 = 4 \int \tau f' \partial_t u \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u) f'^{-3} dt d\theta, \quad (2.14)$$

$$J_3 = \int \left( 2\tau^3 + 2(n-2) \tau^2 f'^{-1} \right) \partial_t (|u|^2) \sqrt{\gamma} dt d\theta, \quad (2.15)$$

Now we will use integration by parts to estimate  $J_i$ . We recall that  $f$  is radial. We find that :

$$\begin{aligned} J_1 &= 4\tau \int f'' |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad - \int 2\tau f' \partial_t \ln \sqrt{\gamma} |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned}$$

Recall that (2.6) implies that  $|\partial_t \ln \sqrt{\gamma}| \leq C e^t$ . Then properties (2.1) on  $f$  gives, for large  $|T_0|$  that  $|\partial_t \ln \sqrt{\gamma}|$  is small compared to  $|f''|$ . Therefore one has

$$J_1 \geq -5\tau \int |f''| \cdot |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \quad (2.16)$$

In order to estimate  $J_2$  we first integrate by parts with respect to  $\partial_i$  :

$$J_2 = -2 \int 2\tau f' \partial_i \partial_i u \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta.$$

Then we integrate by parts with respect to  $\partial_t$ . We get :

$$\begin{aligned} J_2 &= -4\tau \int f'' \gamma^{ij} \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad + \int 2\tau f' \partial_t \ln \sqrt{\gamma} \gamma^{ij} \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad + \int 2\tau f' \partial_t (\gamma^{ij}) \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned}$$

We denote  $|D_\theta u|^2 = \partial_i u \gamma^{ij} \partial_j u$ . Now using that  $-f''$  is non-negative and  $\tau$  is large, the conditions (2.1) and (2.6) gives for  $|T_0|$  large enough:

$$J_2 \geq 3\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \quad (2.17)$$

Similarly computation of  $J_3$  gives :

$$\begin{aligned} J_3 &= - \int (2\tau^3 + 2(n-2)\tau^2 f'^{-1}) \partial_t \ln \sqrt{\gamma} |u|^2 \sqrt{\gamma} dt d\theta \\ &\quad + 2 \int (n-2)\tau^2 f'' f' |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \end{aligned} \quad (2.18)$$

From (2.1) , (2.6) we have provided that  $\tau$  and  $|T_0|$  are large enough :

$$\begin{aligned} J_3 &\geq -3\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad - c\tau^2 \int |f''| \cdot |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \end{aligned} \quad (2.19)$$

Thus far, using (2.16), (2.17) and (2.19), we have :

$$\begin{aligned} I_3 &\geq 4\tau \int |f''| |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad - c\tau \int |f''| |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - c\tau^2 \int |f''| \cdot |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.20)$$

Now we consider  $I_1$  :

$$I_1 = \left\| \partial_t^2 u + (\tau^2 f'^2 + (n-2)\tau f')u + \Delta_\theta u \right\|_f^2.$$

Let  $\rho > 0$  a small number to be chosen later. Since  $|f''| \leq \mu_4$  and  $\tau \geq 1$ , we have :

$$I_1 \geq \frac{\rho}{\tau} I'_1, \quad (2.21)$$

where  $I'_1$  is defined by :

$$I'_1 = \left\| \sqrt{|f''|} \left[ \partial_t^2 u + (\tau^2 f'^2 + (n-2)\tau f')u + \Delta_\theta u \right] \right\|_f^2 \quad (2.22)$$

and one has

$$I'_1 = K_1 + K_2 + K_3, \quad (2.23)$$

with

$$\begin{aligned} K_1 &= \left\| \sqrt{|f''|} (\partial_t^2 u + \Delta_\theta u) \right\|_f^2, \\ K_2 &= \left\| \sqrt{|f''|} (\tau^2 f'^2 + (n-2)\tau f')u \right\|_f^2, \\ K_3 &= 2 \left\langle (\partial_t^2 u + \Delta_\theta u) |f''|, (\tau^2 f'^2 + (n-2)\tau f')u \right\rangle_f. \end{aligned} \quad (2.24)$$

We first estimate  $K_3$ . We set

$$A := A(t) = (\tau^2 f'^2 f'' + (n-2)\tau f' f'').$$

Using properties (2.1), we notice the following estimates

$$|A| \leq C\tau^2 |f''| \quad (2.25)$$

$$|\partial_t A| \leq C\tau^2 |f''| \quad (2.26)$$

Now, recall that  $|f''| = -f''$ , we can write

$$K_3 = -2 \int A(\partial_t^2 u + \Delta_\theta u) u f'^{-3} \sqrt{\gamma} dt d\theta$$

Integrating by parts gives :

$$\begin{aligned} K_3 &= 2 \int A |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &+ 2 \int \partial_t A \partial_t u u f'^{-3} \sqrt{\gamma} dt d\theta \\ &- 6 \int A(t) f'' f'^{-1} \partial_t u u f'^{-3} \sqrt{\gamma} dt d\theta \\ &+ 2 \int A(t) \partial_t \ln \sqrt{\gamma} \partial_t u u f'^{-3} \sqrt{\gamma} dt d\theta \\ &+ 2 \int A(t) |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \end{aligned} \quad (2.27)$$

Now since  $2\partial_t u u \leq u^2 + |\partial_t u|^2$ , we can use properties on  $f$  (2.1),  $\gamma$  (2.6) and  $A$  (2.25, 2.26) to get

$$K_3 \geq -c\tau^2 \int |f''| (|\partial_t u|^2 + |D_\theta u|^2 + |u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \quad (2.28)$$

To estimate  $K_2$  (2.24), we notice that

$$\begin{aligned} K_2 &\geq c_1 \tau^4 \|\sqrt{|f''|} |u|\|_f^2 - c_2 \tau^2 \|\sqrt{|f''|} |u|\|_f^2 \\ K_2 &\geq C\tau^4 \|\sqrt{|f''|} |u|\|_f^2 \end{aligned} \quad (2.29)$$

and since  $K_1 \geq 0$ ,

$$\begin{aligned} I_1 &\geq -\rho c \tau \int |f''| (|\partial_t u|^2 + |D_\theta u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \\ &+ C\tau^3 \rho \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.30)$$

Then using (2.20) and (2.30)

$$\begin{aligned}
I^2 &\geq 4\tau^2 \|f' \partial_t u\|_f^2 + 4\tau \int |f''| |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad + C\tau^3 \rho \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad - \rho c\tau \int |f''| (|u|^2 + |\partial_t u|^2 + |D_\theta u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad - c\tau^2 \int |f''| \cdot |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta.
\end{aligned} \tag{2.31}$$

Now one needs to check that every non-positive terms in the right hand side of (2.31) can be absorbed in the first three terms.

First fix  $\rho$  small enough such that  $\rho \leq \frac{2}{c}$ , where  $c$  is the constant appearing in (2.31), then we obviously have

$$\rho c\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \leq 2\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta.$$

But the following integral

$$-\rho c\tau \int |f''| (|u|^2 + |\partial_t u|^2) f'^{-3} \sqrt{\gamma} dt d\theta,$$

can also be absorbed by comparing powers of  $\tau$  of the positive terms. Finally since conditions (2.1) imply that  $e^t$  is small compared to  $|f''|$ , we can absorb  $-c\tau^3 e^t |u|^2$  in  $C\tau^3 \rho |f''| |u|^2$ .

$$\begin{aligned}
I &\geq C\tau^2 \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C\tau \int |f''| |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad + C\tau^3 \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta
\end{aligned} \tag{2.32}$$

We can now check that  $II$  can be absorbed in  $I$ . Since for  $|T_0|$  and  $\tau$  large enough, we have :

$$II \leq c(\tau^2 \|f'' u\|_f^2 + \|e^t u\|_f^2 + \|\partial_t u\|_f^2 + \|e^t \partial_t u\|_f^2),$$

All this terms can easily be absorbed in (2.32).

Then we obtain

$$\|L_\tau u\|_f^2 \geq C\tau^3 \|\sqrt{|f''|} u\|_f^2 + C\tau^2 \|\partial_t u\|_f^2 + C\tau \|\sqrt{|f''|} |D_\theta u|\|_f^2. \tag{2.33}$$

The condition  $\mu_4 \geq |f''|$  force  $\|\partial_t u\|_f^2 \geq \mu_4^{-1} \|\sqrt{|f''|} \partial_t u\|_f^2$ , then replacing  $\tau^2$  by  $\tau$  in the term involving  $\partial_t u$  we can state that

$$\|L_\tau u\|_f^2 \geq C\tau^3 \|\sqrt{|f''|} u\|_f^2 + C\tau \|\sqrt{|f''|} \partial_t u\|_f^2 + C\tau \|\sqrt{|f''|} |D_\theta u|\|_f^2. \tag{2.34}$$

Then we set  $u = e^{\tau\phi} v$ . We first note that

$$\tau \|\sqrt{|f''|} \partial_t (e^{\tau\phi} v)\|_f^2 \geq \frac{\tau}{2} \|\sqrt{|f''|} e^{\tau\phi} \partial_t v\|_f^2 - \mu_2 \tau^3 \|\sqrt{|f''|} e^{\tau\phi} v\|_f^2$$

We can deduce from the definition of  $\phi$ , for  $C$  large enough that

$$\begin{aligned}
\|e^{2t} e^{\tau\phi} (\Delta v)\|_f^2 &\geq C\tau^3 \|\sqrt{|f''|} e^{\tau\phi} v\|_f^2 \\
+ C\tau \|e^{\tau\phi} \partial_t v\|_f^2 &+ C\tau \|\sqrt{|f''|} e^{\tau\phi} D_\theta v\|_f^2.
\end{aligned} \tag{2.35}$$



It remains to compare  $\|\cdot\|_f$  to the usual  $L^2$  norm. First note that since  $0 < \mu_1 \leq f' \leq \mu_2$ , (2.35) we can drop the term  $(f')^{-3}$  in the integrals of (2.35). Recall that in polar coordinates  $(r, \theta)$  the volume element is  $r^{n-1} \sqrt{\gamma} dr d\theta$  and that  $r = e^t$ , we can deduce from (2.35) that :

$$\begin{aligned} \|r^2 e^{\tau\phi} \Delta v r^{-\frac{n}{2}}\|^2 &\geq C\tau^3 \|\sqrt{|f''|} e^{\tau\phi} v r^{-\frac{n}{2}}\|^2 \\ &+ C\tau \|r \sqrt{|f''|} e^{\tau\phi} |\nabla v| r^{-\frac{n}{2}}\|^2. \end{aligned} \quad (2.36)$$

This achieves the proof of lemma 2.2.  $\square$

*proof of theorem 2.1.* Now we additionally suppose that  $\text{supp}(u) \subset \{x \in M; r(x) \geq \delta > 0\}$  and define  $T_1 = \ln \delta$ .

Cauchy-Schwarz inequality apply to

$$\int \partial_t(u^2) e^{-t} \sqrt{\gamma} dt d\theta = 2 \int u \partial_t u e^{-t} \sqrt{\gamma} dt d\theta$$

gives

$$\int \partial_t(u^2) e^{-t} \sqrt{\gamma} dt d\theta \leq 2 \left( \int (\partial_t u)^2 e^{-t} \sqrt{\gamma} dt d\theta \right)^{\frac{1}{2}} \left( \int u^2 e^{-t} \sqrt{\gamma} dt d\theta \right)^{\frac{1}{2}}. \quad (2.37)$$

On the other hand, integrating by parts gives

$$\int \partial_t(u^2) e^{-t} \sqrt{\gamma} dt d\theta = \int u^2 e^{-t} \sqrt{\gamma} dt d\theta - \int u^2 e^{-t} \partial_t(\ln(\sqrt{\gamma})) \sqrt{\gamma} dt d\theta. \quad (2.38)$$

Now since  $|\partial_t \ln \sqrt{\gamma}| \leq C e^t$  for  $|T_0|$  large enough we can deduce :

$$\int \partial_t(u^2) e^{-t} \sqrt{\gamma} dt d\theta \geq c \int u^2 e^{-t} \sqrt{\gamma} dt d\theta. \quad (2.39)$$

Combining (2.37) and (2.39) gives

$$\begin{aligned} c^2 \int u^2 e^{-t} \sqrt{\gamma} dt d\theta &\leq 4 \int (\partial_t u)^2 e^{-t} \sqrt{\gamma} dt d\theta \\ &\leq 4e^{-T_1} \int (\partial_t u)^2 \sqrt{\gamma} dt d\theta. \end{aligned}$$

Finally, dropping all terms except  $\|2\tau f' \partial_t u\|^2$  in (2.32) gives :

$$C' I^2 \geq \tau^2 \delta^2 \|r^{-1} u\|^2.$$

Inequality (2.32) can then be replaced by :

$$\begin{aligned} I^2 &\geq C_1 \tau \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C_1 \tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &+ C_1 \tau^3 \int |f''| \cdot |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C_1 \tau^2 \delta^2 \int |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.40)$$

Then following the end of the proof of lemma 2.2, we can state that for  $\tau \geq C_2$  with  $C_2$  great enough and depending only on  $(M, g)$  the following estimate holds:

$$\begin{aligned} C \left\| r^2 e^{\tau\phi} \Delta u r^{-n/2} \right\|^2 &\geq \tau^3 \left\| \sqrt{|f''(\ln r)|} e^{\tau\phi} u r^{-n/2} \right\|^2 \\ &+ \tau^2 \delta \left\| r^{-\frac{1}{2}} e^{\tau\phi} u r^{-n/2} \right\|^2 + \tau \left\| r \sqrt{|f''(\ln r)|} e^{\tau\phi} |\nabla u|^2 r^{-n/2} \right\|^2. \end{aligned} \quad (2.41)$$

We will now derive from (2.41), the Carleman estimate on  $\Delta + W$ . If  $W$  is a bounded function, one has from triangular inequality :

$$\begin{aligned} C \|r^2 e^{\tau\phi} (\Delta u + Wu) r^{-\frac{n}{2}}\|^2 &\geq \frac{C}{2} \|r^2 e^{\tau\phi} (\Delta u + Wu) r^{-\frac{n}{2}}\|^2 \\ &\quad - C \|W\|_\infty^2 \cdot \|r^2 e^{\tau\phi} u r^{-\frac{n}{2}}\|^2 \end{aligned}$$

Since for  $R_0$  small enough one has  $r^2 \leq \sqrt{|f''(\ln r)|}$  from (2.1), assuming additionally that  $\tau^3 \geq C_1 \|W\|_\infty^2$ , we can use (2.41) to absorb the negative term and conclude the proof.  $\square$

### 2.1.1 Special choice of weight function

In this paragraph we derive a special case of theorem 2.1 which will be useful for the remaining of this paper. Let  $\varepsilon$  be a real number such that  $0 < \varepsilon < 1$ . As in previous example we consider on  $] -\infty, T_0]$  the function defined by

$$f(t) = t - e^{\varepsilon t}.$$

Therefore one can easily check that

$$\begin{aligned} 1 - \varepsilon &\leq f'(t) \leq 1, \\ \varepsilon^{-1} |f^{(3)}(t)| &\leq -f''(t) \leq \varepsilon^2, \\ \lim_{t \rightarrow -\infty} f''(t) e^{-t} &= +\infty, \end{aligned}$$

and then  $f$  statifies the properties (2.1). Therefore we can apply theorem 2.1 with the weight function  $\phi(x) = -f(\ln r) = -\ln r + r^\varepsilon$ . Now we notice that

$$e^{\tau\phi} = e^{-\tau \ln r} e^{\tau r^\varepsilon}.$$

The point is that we can now obtain a Carleman estimate with the usual  $L^2$  norm. Indeed one has

$$e^{\tau\phi} r^{-\frac{n}{2}} = e^{(\tau + \frac{n}{2})\phi} e^{-\frac{n}{2} r^\varepsilon}$$

and therefore for  $r$  small enough

$$\frac{1}{2} e^{(\tau + \frac{n}{2})\phi} \leq e^{\tau\phi} r^{-\frac{n}{2}} \leq e^{(\tau + \frac{n}{2})\phi}.$$

Then we have

**Corollary 2.4.** *There exist positive constants  $R_0, C, C_1, C_2$ , which depend only on  $M$  and  $\varepsilon$ , such that, for any  $W \in L^\infty(M)$ ,  $x_0 \in M$ , any  $u \in C_0^\infty(B_{R_0}(x_0) \setminus B_\delta(x_0))$  and  $\tau \geq C_1 \|W\|_\infty^{\frac{2}{3}} + C_2$ , one has*

$$\begin{aligned} C \|r^2 e^{\tau\phi} (\Delta u + Wu)\|^2 &\geq \tau^3 \|r^{\frac{\varepsilon}{2}} e^{\tau\phi} u\|^2 \\ + \tau^2 \delta \|r^{-\frac{1}{2}} e^{\tau\phi} u\|^2 &+ \tau \|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla u\|^2. \end{aligned} \quad (2.42)$$

We emphasis that  $\varepsilon$  is fixed and its value will not have any influence on ours statements as long as  $0 < \varepsilon < 1$ . Therefore we may and will omit the dependency of our constants on  $\varepsilon$ .

## 2.2 Three balls inequality

We now want to derive from our Carleman estimate, a control on the local behaviour of solutions. We will first give and prove an Hadamard three circles type theorem. To obtain such result the basic idea is to apply Carleman estimate to  $\chi u$  where  $\chi$  is an appropriate cut off functions and  $u$  a solution of (1.1).

**Proposition 2.5** (Three balls inequality). *There exist positive constants  $R_1, C_1, C_2$  and  $0 < \alpha < 1$  wich depend only on  $(M, g)$  such that, if  $u$  is a solution to (1.1) with  $W$  of class  $L^\infty$ , then for any  $R < R_1$ , and any  $x_0 \in M$ , one has*

$$\|u\|_{B_R(x_0)} \leq e^{C_1 \|W\|_\infty^{2/3} + C_2} \|u\|_{B_{\frac{R}{2}}(x_0)}^\alpha \|u\|_{B_{2R}(x_0)}^{1-\alpha}. \quad (2.43)$$

*Proof.* Let  $x_0$  a point in  $M$ . Let  $u$  be a solution to (1.1) and  $R$  such that  $0 < R < \frac{R_0}{2}$  with  $R_0$  as in corollary 2.4. We will denote by  $\|v\|_{R_1, R_2}$  the  $L^2$  norm of  $v$  on the set  $A_{R_1, R_2} := \{x \in M; R_1 \leq r(x) \leq R_2\}$ . Let  $\psi \in \mathcal{C}_0^\infty(B_{2R})$ ,  $0 \leq \psi \leq 1$ , a function with the following properties:

- $\psi(x) = 0$  if  $r(x) < \frac{R}{4}$  or  $r(x) > \frac{5R}{3}$ ,
- $\psi(x) = 1$  if  $\frac{R}{3} < r(x) < \frac{3R}{2}$ ,
- $|\nabla \psi(x)| \leq \frac{C}{R}$ ,
- $|\nabla^2 \psi(x)| \leq \frac{C}{R^2}$ .

First since the function  $\psi u$  is supported in the annulus  $A_{\frac{R}{3}, \frac{5R}{3}}$ , we can apply estimate (2.42) of theorem 2.4. In particular we have, since the quotient between  $\frac{R}{3}$  and  $\frac{5R}{3}$  don't depend on  $R$  :

$$C \|r^2 e^{\tau \phi} (\Delta \psi u + 2 \nabla u \cdot \nabla \psi)\| \geq \tau \|e^{\tau \phi} \psi u\|. \quad (2.44)$$

Assume that  $\tau \geq 1$ , and use properties of  $\psi$  to get :

$$\begin{aligned} \|e^{\tau \phi} u\|_{\frac{R}{3}, \frac{3R}{2}} &\leq C \left( \|e^{\tau \phi} u\|_{\frac{R}{4}, \frac{R}{3}} + \|e^{\tau \phi} u\|_{\frac{3R}{2}, \frac{5R}{3}} \right) \\ &+ C \left( R \|e^{\tau \phi} \nabla u\|_{\frac{R}{4}, \frac{R}{3}} + R \|e^{\tau \phi} \nabla u\|_{\frac{3R}{2}, \frac{5R}{3}} \right). \end{aligned} \quad (2.45)$$

Recall that  $\phi(x) = -\ln r(x) + r(x)^\varepsilon$ . In particular  $\phi$  is radial and decreasing (for small  $r$ ). Then one has,

$$\begin{aligned} \|e^{\tau \phi} u\|_{\frac{R}{3}, \frac{3R}{2}} &\leq C \left( e^{\tau \phi(\frac{R}{4})} \|u\|_{\frac{R}{4}, \frac{R}{3}} + e^{\tau \phi(\frac{3R}{2})} \|u\|_{\frac{3R}{2}, \frac{5R}{3}} \right) \\ &+ C \left( R e^{\tau \phi(\frac{R}{4})} \|\nabla u\|_{\frac{R}{4}, \frac{R}{3}} + R e^{\tau \phi(\frac{3R}{2})} \|\nabla u\|_{\frac{3R}{2}, \frac{5R}{3}} \right). \end{aligned}$$

Now we recall the following elliptic estimates : since  $u$  satisfies (1.1) then it is not hard to see that :

$$\|\nabla u\|_{(1-a)R} \leq C \left( \frac{1}{(1-a)R} + \|W\|_\infty^{1/2} \right) \|u\|_R, \quad \text{for } 0 < a < 1. \quad (2.46)$$

Moreover since  $A_{R_1, R_2} \subset B_{R_2}$ , using formula (2.46) and properties of  $\phi$  gives

$$e^{\tau \phi(\frac{3R}{2})} \|u\|_{\frac{3R}{2}, \frac{5R}{3}} \leq C \left( \frac{1}{R} + \|W\|_\infty^{1/2} \right) e^{\tau \phi(\frac{3R}{2})} \|u\|_{2R}.$$

Using (2.45) one has :

$$\|u\|_{\frac{R}{3}, R} \leq C(\|W\|_\infty^{1/2} + 1) \left( e^{\tau(\phi(\frac{R}{4}) - \phi(R))} \|u\|_{\frac{R}{2}} + e^{\tau(\phi(\frac{3R}{2}) - \phi(R))} \|u\|_{2R} \right).$$

Let  $A_R = \phi(\frac{R}{4}) - \phi(R)$  and  $B_R = -(\phi(\frac{3R}{2}) - \phi(R))$ . From the properties of  $\phi$ , we have  $0 < A^{-1} \leq A_R \leq A$  and  $0 < B \leq B_R \leq B^{-1}$  where  $A$  and  $B$  don't depend on  $R$ . We may assume that  $C(\|W\|_\infty^{1/2} + 1) \geq 2$ . Then we can add  $\|u\|_{\frac{R}{3}}$  to each side and bound it in the right hand side by  $C(\|W\|_\infty^{1/2} + 1)e^{\tau A} \|u\|_{\frac{R}{2}}$ . We get :

$$\|u\|_R \leq C(\|W\|_\infty^{1/2} + 1) \left( e^{\tau A} \|u\|_{\frac{R}{2}} + e^{-\tau B} \|u\|_{2R} \right). \quad (2.47)$$

Now we want to find  $\tau$  such that

$$C(\|W\|_\infty^{1/2} + 1)e^{-\tau B} \|u\|_{2R} \leq \frac{1}{2} \|u\|_R$$

wich is true for  $\tau \geq -\frac{1}{B} \ln \left( \frac{1}{2C(\|W\|_\infty^{1/2} + 1)} \frac{\|u\|_R}{\|u\|_{2R}} \right)$ . Since  $\tau$  must also satisfy

$$\tau \geq C_1 \|W\|_\infty^{2/3} + C_2,$$

we choose

$$\tau = -\frac{1}{B} \ln \left( \frac{1}{2C(\|W\|_\infty^{1/2} + 1)} \frac{\|u\|_R}{\|u\|_{2R}} \right) + C_1 \|W\|_\infty^{2/3} + C_2. \quad (2.48)$$

Up to a change of constant we may assume that  $C(\|W\|_\infty^{1/2} + 1) \leq C_1 \|W\|_\infty^{2/3} + C_2$  then we can deduce from (2.47) that :

$$\|u\|_{\frac{R+B}{B}} \leq e^{C_1 \|W\|_\infty^{2/3} + C_2} \|u\|_{\frac{A}{2R}} \|u\|_{\frac{R}{2}}, \quad (2.49)$$

Finally define  $\alpha = \frac{A}{A+B}$  and taking exponent  $\frac{B}{A+B}$  of (2.49) gives the result.  $\square$

### 2.3 Doubling estimates

Now we intend to show that the vanishing order of solutions to (1.1) is everywhere bound by  $C_1 \|W\|_\infty^{2/3} + C_2$ . This is an immediate consequence of the following :

**Theorem 2.6** (doubling estimate). *There exist two positive constants  $C_1$  and  $C_2$ , depending only on  $M$  such that : if  $u$  is a solution to (1.1) on  $M$  with  $W$  of class  $\mathcal{C}^1$  then for any  $x_0$  in  $M$  and any  $r > 0$ , one has*

$$\|u\|_{B_{2r}(x_0)} \leq e^{C_1 \|W\|_\infty^{2/3} + C_2} \|u\|_{B_r(x_0)}. \quad (2.50)$$

**Remark 2.7.** Using standard elliptic theory to bound the  $L^\infty$  norm of  $|u|$  by a multiple of its  $L^2$  norm on a greater ball (see by example [5] Theorem 8.17 and problem 8.3), show that the doubling estimate is still true with the  $L^\infty$  norm :

$$\|u\|_{L^\infty(B_{2r}(x_0))} \leq e^{C_1 \|W\|_\infty^{2/3} + C_2} \|u\|_{L^\infty(B_r(x_0))}. \quad (2.51)$$

To prove the theorem 2.6 we need to use the standard overlapping chains of balls argument ([3, 9, 13]) to show :

**Proposition 2.8.** *For any  $R > 0$  there exists  $C_R > 0$  such that for any  $x_0 \in M$ , any  $W \in \mathcal{C}^1(M)$  and any solutions  $u$  to (1.1) :*

$$\|u\|_{B_R(x_0)} \geq e^{-C_R(1+\|W\|_\infty^{2/3})} \|u\|_{L^2(M)}.$$

*Proof.* We may assume without loss of generality that  $R < R_0$ , with  $R_0$  as in the three balls inequality (proposition 2.43). Up to multiplication by a constant, we can assume that  $\|u\|_{L^2(M)} = 1$ . We denote by  $\bar{x}$  a point in  $M$  such that  $\|u\|_{B_R(\bar{x})} = \sup_{x \in M} \|u\|_{B_R(x)}$ . This implies that one has  $\|u\|_{B_R(\bar{x})} \geq D_R$ , where  $D_R$  depend only on  $M$  and  $R$ . One has from proposition (2.43) at an arbitrary point  $x$  of  $M$  :

$$\|u\|_{B_{R/2}(x)} \geq e^{-c(1+\|W\|_\infty^{2/3})} \|u\|_{B_R(\bar{x})}^{\frac{1}{\alpha}}. \quad (2.52)$$

Let  $\gamma$  be a geodesic curve between  $x_0$  and  $\bar{x}$  and define  $x_1, \dots, x_m = \bar{x}$  such that  $x_i \in \gamma$  and  $B_{\frac{R}{2}}(x_{i+1}) \subset B_R(x_i)$ , for any  $i$  from 0 to  $m-1$ . The number  $m$  depends only on  $\text{diam}(M)$  and  $R$ . Then the properties of  $(x_i)_{1 \leq i \leq m}$  and inequality (2.52) give for all  $i$ ,  $1 \leq i \leq m$  :

$$\|u\|_{B_{R/2}(x_i)} \geq e^{-c(1+\|W\|_\infty^{2/3})} \|u\|_{B_{R/2}(x_{i+1})}^{\frac{1}{\alpha}}. \quad (2.53)$$

The result follows by iteration and the fact that  $\|u\|_{B_R(\bar{x})} \geq D_R$ .  $\square$

**Corollary 2.9.** *For all  $R > 0$ , there exists a positive constant  $C_R$  depending only on  $M$  and  $R$  such that at any point  $x_0$  in  $M$  one has*

$$\|u\|_{B_{R,2R}} \geq e^{-C_R(1+\|W\|_\infty^{2/3})} \|u\|_{L^2(M)}.$$

*Proof.* Recall that  $\|u\|_{B_{R,2R}} = \|u\|_{L^2(A_{R,2R})}$  with  $A_{R,2R} := \{x; R \leq d(x, x_0) \leq 2R\}$ . Let  $R < R_0$  where  $R_0$  is from proposition 2.45, note that  $R_0 \leq \text{diam}(M)$ . Since  $M$  is geodesically complete, there exists a point  $x_1$  in  $A_{R,2R}$  such that  $B_{x_1}(\frac{R}{4}) \subset A_{R,2R}$ . From proposition 2.8 one has

$$\|u\|_{B_{\frac{R}{4}}(x_1)} \geq e^{-C_R(1+\|W\|_\infty^{2/3})} \|u\|_{L^2(M)}$$

wich gives the result.  $\square$

*Proof of theorem 2.6.* We proceed as in the proof of three balls inequality (proposition 2.45) except for the fact that now we want the first ball to become arbitrary small in front of the others. Let  $R = \frac{R_0}{4}$  with  $R_0$  as in the three balls inequality, let  $\delta$  such that  $0 < 3\delta < \frac{R}{8}$ , and define a smooth function  $\psi$ , with  $0 \leq \psi \leq 1$  as follows:

- $\psi(x) = 0$  if  $r(x) < \delta$  or if  $r(x) > R$ ,
- $\psi(x) = 1$  if  $r(x) \in [\frac{5\delta}{4}, \frac{R}{2}]$ ,
- $|\nabla \psi(x)| \leq \frac{C}{\delta}$  and  $|\nabla^2 \psi(x)| \leq \frac{C}{\delta^2}$  if  $r(x) \in [\delta, \frac{5\delta}{4}]$ ,

- $|\nabla\psi(x)| \leq C$  and  $|\nabla^2\psi(x)| \leq C$  if  $r(x) \in [\frac{R}{2}, R]$ .

Keeping appropriate terms in (2.42) applied to  $\psi u$  gives :

$$\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}\psi u\| + \tau\delta^{\frac{1}{2}}\|r^{-\frac{1}{2}}e^{\tau\phi}\psi u\| \leq C \left( \|r^2e^{\tau\phi}\nabla u \cdot \nabla\psi\| + \|r^2e^{\tau\phi}\Delta\psi u\| \right).$$

Using properties of  $\psi$ , one has

$$\begin{aligned} \|r^{\frac{\varepsilon}{2}}e^{\tau\phi}u\|_{\frac{R}{8}, \frac{R}{4}} + \|e^{\tau\phi}u\|_{\frac{5\delta}{4}, 3\delta} &\leq C \left( \delta\|e^{\tau\phi}\nabla u\|_{\delta, \frac{5\delta}{4}} + \|e^{\tau\phi}\nabla u\|_{\frac{R}{2}, R} \right) \\ &+ C \left( \|e^{\tau\phi}u\|_{\delta, \frac{5\delta}{4}} + \|e^{\tau\phi}u\|_{\frac{R}{2}, R} \right). \end{aligned}$$

Then (2.46) and properties of  $\phi$ , we get

$$\begin{aligned} e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{8}, \frac{R}{4}} + e^{\tau\phi(3\delta)}\|u\|_{\frac{5\delta}{4}, 3\delta} \\ \leq C(1 + \|W\|_{\infty}^{1/2}) \left( e^{\tau\phi(\delta)}\|u\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{3})}\|u\|_{\frac{5R}{3}} \right), \end{aligned}$$

and adding  $e^{\tau\phi(3\delta)}\|u\|_{\frac{5\delta}{4}}$  to each side leads to

$$\begin{aligned} e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{8}, \frac{R}{4}} + e^{\tau\phi(3\delta)}\|u\|_{3\delta} \\ \leq C(1 + \|W\|_{\infty}^{1/2}) \left( e^{\tau\phi(\delta)}\|u\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{3})}\|u\|_{\frac{5R}{3}} \right). \end{aligned}$$

Now we want to choose  $\tau$  such that

$$C(1 + \|W\|_{\infty}^{1/2})e^{\tau\phi(\frac{R}{3})}\|u\|_{\frac{5R}{3}} \leq \frac{1}{2}e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{8}, \frac{R}{4}}.$$

For the same reasons than before we choose

$$\tau = \frac{1}{\phi(\frac{R}{3}) - \phi(\frac{R}{4})} \ln \left( \frac{1}{2C(1 + \|W\|_{\infty}^{1/2})} \frac{\|u\|_{\frac{R}{8}, \frac{R}{4}}}{\|u\|_{\frac{5R}{3}}} \right) + C_1(1 + \|W\|_{\infty}^{2/3}).$$

Define  $D_R = (\phi(\frac{R}{3}) - \phi(\frac{R}{4}))^{-1}$ ; like before one has  $0 < A^{-1} \leq D_R \leq A$ . Dropping the first term in the left hand side, one has

$$\|u\|_{3\delta} \leq e^{C(1+\|W\|_{\infty}^{2/3})} \left( \frac{\|u\|_{\frac{R}{8}, \frac{R}{4}}}{\|u\|_{\frac{5R}{3}}} \right)^A \|u\|_{\frac{3\delta}{2}}$$

Finally from corollary 2.9, define  $r = \frac{3\delta}{2}$  to have :

$$\|u\|_{2r} \leq e^{C(1+\|W\|_{\infty}^{2/3})}\|u\|_r.$$

Thus, the theorem is proved for all  $r \leq \frac{R_0}{16}$ . Using proposition 2.8 we have for  $r \geq \frac{R_0}{16}$  :

$$\begin{aligned} \|u\|_{B_{x_0}(r)} &\geq \|u\|_{B_{x_0}(\frac{R_0}{16})} \geq e^{-C_0(1+\|W\|_{\infty}^{2/3})}\|u\|_{L^2(M)} \\ &\geq e^{-C_1(1+\|W\|_{\infty}^{2/3})}\|u\|_{B_{x_0}(2r)}. \end{aligned}$$

□

Finally theorem 1.1 is an easy and direct consequence of this doubling estimates.

### 3 Possible vanishing order for solutions

The aim of this section is to prove theorem 1.2 which states that our exponent  $\frac{2}{3}$  on  $\|W\|_\infty$  in theorem 1.1 is sharp. More precisely, we will construct a sequence  $(u_k, W_k)$  verifying  $\Delta u_k + W_k u_k = 0$  and such that :

- In at least one point of  $\mathbb{S}^2$ , the function  $u_k$  vanishes at an order at least  $C_1 \|W_k\|_\infty^{\frac{2}{3}} + C_2$ , where  $C_1$  and  $C_2$  are fixed numbers which don't depend on  $k$ .
- $\lim_{k \rightarrow +\infty} \|W_k\|_\infty = +\infty$ .
- The potential  $W$  has compact support and

$$\text{supp}(W) \subset (\mathbb{S}^2 \setminus (\{P\} \cup \{Q\}))$$

Our construction will be inspired by the previous work of Meshkov. In [15], he has shown that one can find non trivial, complex valued, solutions of (1.1) in  $\mathbb{R}^n$  ( $n \geq 2$ ), with the following exponential decay at infinity :  $|u(x)| \leq e^{-C|x|^{\frac{4}{3}}}$ . This gives a negative answer to a question of Landis, [14]. He has also shown that  $\frac{4}{3}$  is the greatest exponent for which one can hope to find non trivial solutions to (1.1) in  $\mathbb{R}^n$ .

Our main point is to construct on  $\mathbb{R}^2$  a solution  $u$  to  $\Delta u + Wu = 0$  with appropriate decay at infinity with respect to the  $L^\infty$  norm of  $W$ . Then, we use stereographic projection to obtain theorem 1.2. This suggests in particular that we not only need the function  $W$  to be bounded, since we have to pull it back to the sphere, but we need the stronger condition

$$|W(x, y)| \leq \frac{C}{1 + |x|^2 + |y|^2}.$$

It actually appears that we can obtain  $W$  with compact support. We recall that our construction relies crucially on the fact that  $W$  and  $u$  are complex valued functions.

From now on, we will denote by  $(r, \varphi)$  polar coordinates of a point in  $\mathbb{R}^2$  and by  $[a, b]$  the annulus  $[a, b] := \{(r, \varphi) \in \mathbb{R}^+ \times [0, 2\pi[, a \leq r \leq b\}$ . We have the following on  $\mathbb{R}^2$ :

**Proposition 3.1.** *Let  $\rho > 0$  a real number and  $N$  an integer. For  $\rho$  and  $N$  great enough there exists  $u$  and  $W$  verifying  $\Delta u + Wu = 0$  on  $\mathbb{R}^2$  such that :*

- $C_2 |x|^{-N} \leq |u(x)| \leq C_1 |x|^{-N}$
- $\|W\|_{L^\infty} \leq CN^{\frac{3}{2}}$
- $W$  has support in  $[\frac{\rho}{2}, \rho + 6]$

*Proof.*

Let  $\rho \geq 1$ , without loss of generality we suppose that  $N$  is the square of an integer. Let define  $\delta = \frac{1}{\sqrt{N}}$ . For  $j$  an integer from 10 to  $\sqrt{N}$ , we note  $n_j = j^2$  and  $k_j = 2j + 1$  and  $\rho_j = \rho + 6(j - 1)\delta$ . On each annulus  $A_j = [\rho_j, \rho_{j+1}]$  we

construct a solution of  $\Delta u + Wu = 0$  such that  $|u| = a_j r^{-n_j}$  on the sphere  $\{\rho_j\}$  and  $|u| = a_{j+1} r^{-n_j - k_j} = a_{j+1} r^{-n_{j+1}}$  on the sphere  $\{\rho_{j+1}\}$ . We set  $a_{10} = 1$ , the numbers  $a_j$ , for  $j > 10$ , still have to be defined. When working in the annulus  $A_j$ , we will write  $a, n, k, \rho$  instead of  $a_j, n_j, k_j, \rho_j$ . Before proceeding to the proof it is useful to notice that

$$\begin{aligned} k &\leq C_1 \sqrt{n} \leq C_1 \sqrt{N} \\ \frac{n}{k} &\leq C_2 \sqrt{N} \\ \frac{1}{\delta} &= \sqrt{N} \\ k\delta &\leq 3 \end{aligned} \tag{3.1}$$

To build  $(u, W)$  in  $A_j$ , it is convenient to divide the construction into four steps corresponding to the four annulus  $[\rho, \rho + 2\delta], [\rho + 2\delta, \rho + 3\delta], [\rho + 3\delta, \rho + 4\delta], [\rho + 4\delta, \rho + 6\delta]$ .

**Step I. Construction on the annulus  $[\rho, \rho + 2\delta]$**

Define

$$u_1 = ar^{-n} e^{-in\varphi},$$

and

$$u_2 = -br^{-n+2k} e^{iF(\varphi)},$$

with  $b = a(\rho + \delta)^{-2k}$  such that  $\left| \frac{u_2(\rho + \delta)}{u_1(\rho + \delta)} \right| = 1$ , the function  $F(\varphi)$  will be made explicit later. For  $m$  from 0 to  $2n + 2k - 1$ , we set  $\varphi_m = \frac{2\pi m}{2n+2k}$  and  $T = \frac{\pi}{n+k}$ .

We will need the following

**Lemma 3.2** ([15], p. 350). *Their exists a real function  $h$  of class  $\mathcal{C}^2$ , periodic of period  $T$ , with the following properties :*

$$\bullet |h(\varphi)| \leq 5kT, \quad \forall \varphi \in \mathbb{R}, \tag{3.2}$$

$$\bullet |h'(\varphi)| \leq 5k, \quad \forall \varphi \in \mathbb{R}, \tag{3.3}$$

$$\bullet |h''(\varphi)| \leq Ckn, \quad \forall \varphi \in \mathbb{R}, \tag{3.4}$$

$$\bullet h(\varphi) = -4k(\varphi - \varphi_m), \quad \text{for } \varphi_m - \frac{T}{5} \leq \varphi \leq \varphi_m + \frac{T}{5}. \tag{3.5}$$

Let now choose  $F$  as follows

$$F(\varphi) = (n + 2k)\varphi + h(\varphi). \tag{3.6}$$

We also consider two smooths radial functions  $\psi_1, \psi_2$  with the following properties :

$$\bullet 0 \leq \psi_i \leq 1, \tag{3.7}$$

$$\bullet |\psi_i^{(p)}| \leq C\delta^{-p}, \tag{3.8}$$

$$\bullet \psi_1 = 1 \text{ on } [\rho, \rho + \frac{5}{3}\delta] \text{ and } \psi_1 = 0 \text{ on } [\rho + (2 - \frac{1}{10})\delta, \rho + 2\delta], \tag{3.9}$$



- $\psi_2 = 0$  on  $[\rho, \rho + \frac{1}{10}\delta]$  and  $\psi_2 = 1$  on  $[\rho + \frac{1}{3}\delta, \rho + 2\delta]$ . (3.10)

Finally we set  $u = \psi_1 u_1 + \psi_2 u_2$ . The estimate on  $[\rho, \rho + 2\delta]$  is divided among four cases.

*Step I.a. The set  $[\rho, \rho + \frac{\delta}{3}]$*

We have  $u = u_1 + \psi_2 u_2$ , and :  $|\frac{u_2}{u_1}| = \left(\frac{r}{\rho + \delta}\right)^{2k}$ .

Then we introduce  $\alpha$  as an upper bound of  $|\frac{u_2}{u_1}|$  :

$$\left|\frac{u_2}{u_1}\right| \leq \left(\frac{\rho + \frac{\delta}{3}}{\rho + \delta}\right)^{2k} := \alpha < 1.$$

We notice that

$$|u| \geq |u_1| - |\psi_2 u_2| \geq |u_1| - |u_2| \geq \frac{1 - \alpha}{\alpha} |u_2|,$$

Since  $\alpha = e^{2k \ln(1 - \frac{2\delta}{3(\rho + \delta)})}$ , a computation of  $\frac{\alpha}{1 - \alpha}$  leads to the following inequality:

$$|u_2| \leq \frac{3(\rho + \delta)}{4k\delta} |u|. \quad (3.11)$$

Now we compute  $\Delta u$  :

$$\Delta u = \psi_2 \Delta u_2 + 2 \frac{d\psi_2}{dr} \frac{\partial u_2}{\partial r} + \left( \frac{1}{r} \frac{d\psi_2}{dr} + \frac{d^2 \psi_2}{dr^2} \right) u_2. \quad (3.12)$$

First we estimate  $\Delta u_2$ . One has

$$\Delta u_2 = \frac{1}{r^2} ((-n + 2k)^2 + iF'' - (F')^2) u_2.$$

From the definition of  $F$  (3.6) this gives :

$$\Delta u_2 = \frac{1}{r^2} [-8kn + ih'' - 2(n + 2k)h' - (h')^2] u_2. \quad (3.13)$$

Using properties of  $h$ , we obtain

$$|\Delta u_2| \leq \frac{1}{r^2} (8kn + Ckn + C_2 k^2) |u_2| \quad (3.14)$$

Then from inequality (3.11) we have ,

$$|\Delta u_2| \leq \frac{1}{r^2} Ckn \frac{3(\rho + \delta)}{k\delta} |u|,$$

Now (3.1) and  $\frac{\rho + \delta}{r^2} \leq 1$  gives:

$$|\Delta u_2| \leq C \frac{n}{\delta} |u| \leq N^{3/2} |u|$$

Now we estimate the other terms in the right and side of (3.12). The conditions on  $\psi$  (3.8) and estimate of  $u_2$  (3.11) lead to

$$\left| \frac{\partial \psi_2}{\partial r} \frac{\partial u_2}{\partial r} \right| \leq \frac{Cn}{k\delta^2} |u|. \quad (3.15)$$

Similarly, we have

$$\left| \left( \frac{d\psi_2}{dr} + \frac{d^2\psi_2}{dr^2} \right) \frac{1}{r} u_2 \right| \leq C \frac{1}{\delta^3} |u| \quad (3.16)$$

Finally,

$$|\Delta u| \leq C \left( \frac{n}{\delta} + \frac{n}{k\delta^2} + \frac{1}{\delta^3} \right) |u|.$$

Then we have shown, using relations (3.1) between  $k$ ,  $n$ ,  $\delta$  and  $N$  that :

$$|\Delta u| \leq CN^{\frac{3}{2}} |u|. \quad (3.17)$$

*Step I.b.*  $[\rho + \frac{5}{3}\delta, \rho + 2\delta]$

We want to estimate  $\frac{|\Delta u|}{|u|}$ , with  $u = u_2 + \psi_1 u_1$ . First since,

$$\frac{|u_2|}{|u_1|} = \frac{b}{a} r^{2k} \geq \left( \frac{\rho + \frac{5}{3}\delta}{\rho + \delta} \right)^{2k} = \beta > 1,$$

we have

$$|u| \geq |u_2| - |\psi_1 u_1| \geq |u_2| - |u_1| \geq \left( \frac{\beta - 1}{\beta} \right) |u_2| > 0. \quad (3.18)$$

One also has

$$\Delta u = \Delta u_2 + 2 \frac{d\psi_1}{dr} \frac{\partial u_1}{\partial r} + \left( \frac{1}{r} \frac{d\psi_1}{dr} + \frac{d^2\psi_1}{dr^2} \right) u_1. \quad (3.19)$$

First since  $r \geq \rho \geq 1$  one has from (3.18)

$$|\Delta u_2| \leq \frac{Ckn}{r^2} |u_2| \leq Ckn \frac{\beta}{\beta - 1} |u|. \quad (3.20)$$

Then we estimate  $\frac{\beta}{\beta - 1}$ . On the real set  $[0, \frac{4}{3}]$ , we have :

$$\begin{aligned} \frac{(3/4) \ln(4x/3)}{1+x} &\leq \frac{\ln(1+x)}{e^x} \leq \frac{x}{1 + 3/4(e^{4/3} - 1)x} \end{aligned}$$

Now from the definition of  $\beta$  and since  $k\delta \leq 3$  this estimates gives,

$$\frac{\beta}{\beta - 1} \leq \frac{C}{k\delta},$$

then from (3.20) we have :

$$|\Delta u_2| \leq C \frac{n}{\delta} |u|. \quad (3.21)$$

Moreover we also have

$$\left| \frac{\partial u_1}{\partial r} \frac{d\psi_1}{dr} \right| \leq \frac{C}{\delta} \left| \frac{\partial u_1}{\partial r} \right|,$$

with

$$\left| \frac{\partial u_1}{\partial r} \right| \leq Cn|u_1| \leq C \frac{n}{\beta-1} |u| \leq \frac{Cn}{k\delta} |u|,$$

thus

$$\left| \frac{\partial u_1}{\partial r} \frac{d\psi_1}{dr} \right| \leq C \frac{n}{k\delta^2} |u|.$$

This also gives

$$\left| \frac{\partial u_1}{\partial r} \frac{d\psi_1}{dr} \right| \leq CN^{\frac{3}{2}} |u|. \quad (3.22)$$

Similarly we have,

$$\begin{aligned} \left| \frac{d^2\psi_1}{dr^2} + \frac{1}{r} \frac{d\psi_1}{dr} \right| |u_1| &\leq C \left( \frac{1}{\delta^2} + \frac{1}{\delta} \right) |u_1| \\ &\leq C \frac{1}{\delta^2} |u_1| \\ &\leq CN^{\frac{3}{2}} |u|. \end{aligned}$$

Here again we have shown

$$|\Delta u| \leq CN^{\frac{3}{2}} |u|.$$

*Step I.c*  $\{(r, \varphi); r \in [\rho + \frac{1}{3}\delta, \rho + \frac{5}{3}\delta], \varphi_m + \frac{T}{5} \leq \varphi \leq \varphi_m + \frac{4T}{5}\}$

On this set we have  $u = u_1 + u_2$  and

$$|u| = |u_2| \left| e^{i(F(\varphi)+n\varphi)} - \frac{a}{br^{2k}} \right|. \quad (3.23)$$

Let  $S(\varphi) = F(\varphi) + n\varphi$ , we have  $S(\varphi_m) = 2\pi m$  and

$$S'(\varphi) = 2n + 2k + h'(\varphi)$$

Since  $|h'(\varphi)| \leq 5k$  we have

$$S'(\varphi) \geq 2n - 3k$$

Now since  $n = j^2$  and  $k = 2j + 1$ , the condition  $j \geq 10$  force  $S'(\varphi) > n$  for  $\varphi \in [\varphi_m, \varphi_{m+1}]$ . Then for

$$\varphi_m + \frac{T}{5} \leq \varphi \leq \varphi_{m+1} - \frac{T}{5}$$

we have

$$S(\varphi) \geq S(\varphi_m + \frac{T}{5}) \geq S(\varphi_m) + \frac{nT}{5}$$

Using the same argument to get an upper bound on  $S(\varphi)$  we can state that

$$2\pi m + \frac{nT}{5} \leq S(\varphi) \leq 2\pi(m+1) - \frac{nT}{5}$$

This can be written

$$2\pi m + \frac{n\pi}{5(n+k)} \leq S(\varphi) \leq 2\pi(m+1) - \frac{n\pi}{5(n+k)}$$

Now since  $j \geq 10$ , we clearly have  $\frac{n\pi}{5(n+k)} \geq \frac{\pi}{7}$ . Then we can state that

$$\left| e^{iS(\varphi)} - \lambda \right| \geq \sin\left(\frac{\pi}{7}\right),$$

for any real number  $\lambda$ . This leads to

$$|u| \geq \sin\left(\frac{\pi}{7}\right)|u_2|. \quad (3.24)$$

Finally using that  $\Delta u = \Delta u_2$ , (3.14) and (3.24) we have

$$\begin{aligned} |\Delta u| = |\Delta u_2| &\leq Ckn|u_2| \leq ckn|u| \\ |\Delta u| &\leq CN^{\frac{3}{2}}|u| \end{aligned} \quad (3.25)$$

$$\text{Step I.d) } \{r \in [\rho + \frac{1}{3}\delta, \rho + \frac{5}{3}\delta], |\varphi - \varphi_m| < \frac{T}{5}\}$$

Here we just need to notice that  $u_2 = -br^{-n+2k}e^{i(n-2k)\varphi}e^{4k\varphi_m}$  is harmonic and that  $\psi_1 = \psi_2 = 1$ . Then  $u$  is harmonic and we simply set  $W = 0$ .

## Step II. Construction on the annulus $[\rho + 2\delta, \rho + 3\delta]$ .

Recall that  $u_2 = -br^{-n+2k}e^{iF(\varphi)}$ . We now define

$$u_3 = -br^{-n+2k}e^{i(n+2k)\varphi}.$$

To pass from  $u_2$  to  $u_3$ , we consider a smooth, radial function  $\psi$  on  $[\rho + 2\delta, \rho + 3\delta]$  with the following properties :

$$\bullet \quad 0 \leq \psi(r) \leq 1 \quad (3.26)$$

$$\bullet \quad \psi(r) = 1 \text{ if } r \leq \rho + \frac{7}{3}\delta \quad (3.27)$$

$$\bullet \quad \psi(r) = 0 \text{ if } r \geq \rho + \frac{8}{3}\delta \quad (3.28)$$

$$\bullet \quad |\psi^{(p)}(r)| \leq \frac{C}{\delta^p} \quad (3.29)$$

Then we set on  $[\rho + 2\delta, \rho + 3\delta]$ ,

$$u = -br^{-n+2k}e^{i[\psi(r)h(\varphi) + (n+2k)\varphi]}$$

with  $h$ , the function of lemma 3.2.

Now we prepare the computaiton of  $\Delta u$ , one has:

$$\begin{aligned} \frac{\partial u}{\partial r} &= \left( \frac{-n+2k}{r} + i\psi'(r)h(\varphi) \right) u \\ \frac{\partial^2 u}{\partial r^2} &= \frac{(-n+2k)(-n+2k-1)}{r^2} u \\ &\quad + 2i \frac{-n+2k}{r} \psi'(r)h(\varphi)u + i\psi''(r)h(\varphi)u - \psi'(r)^2 h(\varphi)^2 u \\ \frac{\partial^2 u}{\partial \varphi^2} &= i\psi(r)h''(\varphi)u + \left( i\psi(r)h'(\varphi) + (n+2k) \right)^2 u \end{aligned}$$

$$\begin{aligned}
\Delta u &= \left( \frac{(-n+2k)(-n+2k-1)}{r^2} + 2i \frac{-n+2k}{r} \psi'(r) h(\varphi) \right) u \\
&+ \left( \frac{-n+2k}{r^2} + \frac{i\psi'(r)h(\varphi)}{r} + i\psi''(r)h(\varphi) - \psi'(r)^2 h(\varphi)^2 \right) u \\
&+ \left( \frac{i\psi(r)h''(\varphi)}{r^2} + \frac{(i\psi(r)h'(\varphi) + (n+2k))^2}{r^2} \right) u
\end{aligned}$$

We get

$$\begin{aligned}
\Delta u &= \left[ \left( \frac{-n+2k}{r} + i\psi'(r)h(\varphi) \right)^2 + ih(r) \left( \frac{\psi'(r)}{r} + \psi''(r) \right) \right] u \\
&+ \left[ \frac{i\psi(r)h''(\varphi)}{r^2} - \frac{(n+2k + \psi(r)h'(\varphi))^2}{r^2} \right] u
\end{aligned}$$

After simplification, the main point is that there is no term of order  $n^2$  left, we obtain :

$$|\Delta u| \leq C(kn + k^2 + |\psi' h|^2 + n|\psi' h| + |h\psi''| + |\psi h''| + n|\psi||h'| + |\psi^2||h'|^2)u$$

Now using (3.1), the properties of  $\psi$  (3.26-3.29) and of  $h$  (3.2-3.5), we have

$$|\Delta u| \leq CN^{\frac{3}{2}}|u|.$$

**Step III. Construction on the annulus  $[\rho + 3\delta, \rho + 4\delta]$ .**

Recall that

$$u_3 = -br^{-n+2k}e^{i(n+2k)\varphi}.$$

In this step, we want to pass from  $u_3$  to the harmonic function:

$$u_4 = -b_1r^{-n-2k}e^{i(n+2k)\varphi}.$$

Let  $d = (\rho + 3\delta)^{4k}$ ,  $b_1 = bd$  and

$$g(r) = dr^{-4k} = \left( \frac{\rho + 3\delta}{r} \right)^{4k}.$$

Then  $|g^p(r)| \leq C^p k^p$ , futhermore

$$g(r) \geq \left( \frac{\rho + 3\delta}{\rho + 4\delta} \right)^{4k} \geq e^{4k \ln(1 - \frac{\delta}{\rho + 4\delta})}$$

We have  $0 \leq \frac{\delta}{\rho + 4\delta} \leq \frac{1}{4}$  and  $\ln(1-x) \geq 4 \ln(\frac{3}{4})x$  for  $x$  in the real set  $[0, \frac{1}{4}]$ , then

$$g(r) \geq e^{-16k \ln(\frac{4}{3}) \frac{k\delta}{\rho + 4\delta}} \geq e^{-16 \ln \frac{4}{3}} \geq e^{-5}$$

We know consider a smooth radial function  $\psi$ , with the following properties :

- $0 \leq \psi \leq 1$
- $|\psi^{(p)}| \leq \frac{C}{\delta^p}$
- $\psi(r) = 1$  if  $r \in [\rho + 3\delta, \rho + (3 + \frac{1}{3})\delta]$
- $\psi(r) = 0$  if  $r \in [\rho + (3 + \frac{2}{3})\delta]$

We let  $f(r) = \psi(r) + (1 - \psi(r))g(r)$  so one has

$$e^{-5} \leq f(r) \leq 1.$$

A computation of  $f'(r)$  and  $f^{(2)}(r)$  leads to the following estimates:

$$\begin{aligned} |f'(r)| &\leq C \left( |\psi'(r)| + |\psi'(r)g(r)| + |\psi g'| \right) \\ |f'(r)| &\leq \frac{C}{\delta} + Ck \leq \frac{C}{\delta}, \end{aligned}$$

and

$$|f''(r)| \leq \frac{C}{\delta^2}.$$

We finally set  $u = u_3 f$ , wich implies that  $|u| > 0$  and  $|u_3| \leq C|u|$ . To compute  $\Delta u_3$  we observe that

$$\begin{aligned} \frac{\partial u_3}{\partial r} &= \frac{-n + 2k}{r} u_3, \\ \frac{\partial^2 u_3}{\partial r^2} &= \frac{(-n + 2k)(-n + 2k - 1)}{r^2} u_3, \\ &\text{and} \\ \frac{\partial^2 u_3}{\partial \varphi^2} &= -(n + 2k)^2 u_3. \end{aligned}$$

Then we have :

$$\Delta u_3 = \frac{1}{r^2} \left( (-n + 2k)(-n + 2k - 1) + (-n + 2k) - (n + 2k)^2 \right) u_3$$

Thus we have the following estimates

$$|\Delta u_3| \leq Ckn|u_3| \leq Ckn|u| \tag{3.30}$$

From  $\left| \frac{\partial u_3}{\partial r} \right| \leq Cnu_3$ , and  $\left| \frac{d^p f}{dr^p} \right| \leq \frac{C}{\delta^p}$  one can deduce that:

$$2 \left| \frac{df}{dr} \frac{\partial u_3}{\partial r} \right| \leq C \frac{n}{\delta} u_3 \leq C \frac{n}{\delta} u,$$

and

$$\left| \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} \right| |u_3| \leq \frac{C}{\delta^2} |u_3| \leq \frac{C}{\delta^2} |u|.$$

We then have the following estimate on  $\Delta u$  :

$$\begin{aligned} |\Delta u| &\leq C \left( \frac{n}{\delta} + kn + \frac{1}{\delta^2} \right) |u| \\ |\Delta u| &\leq CN^{\frac{3}{2}} |u| \end{aligned}$$

**Step IV. Construction on the annulus  $[\rho + 4\delta, \rho + 6\delta]$**

Recall that

$$u_4 = -b_1 r^{-(n+2k)} e^{i(n+2k)\varphi}$$

and let

$$u_5 = cr^{-n-k} e^{-i(n+k)\varphi}. \quad (3.31)$$

We choose  $c$  such that

$$\left| \frac{u_5(\rho + 5\delta)}{u_4(\rho + 5\delta)} \right| = 1 \quad (3.32)$$

then

$$\left| \frac{u_5}{u_4} \right| = \frac{r^k}{(\rho + 5\delta)^k}.$$

Like step I we consider two smooth radial functions  $\psi_4$  and  $\psi_5$  with the following properties

- $\psi_4 = 1$  on  $[\rho + 4\delta, \rho + (4 + \frac{5}{3})\delta]$  and  $\psi_4 = 0$  on  $[\rho + (5, 9)\delta, \rho + 6\delta]$
- $\psi_5 = 0$  on  $[\rho + 4\delta, \rho + (4, 1)\delta]$  and  $\psi_5 = 1$  on  $[\rho + (4 + \frac{1}{3})\delta, \rho + 6\delta]$

We let

$$u = \psi_4 u_4 + \psi_5 u_5.$$

Now we estimate  $\Delta u$ . Here again it is convenient to divide this into three steps:

*Step IV.a*  $[\rho + (4 + \frac{1}{3})\delta, \rho + (4 + \frac{5}{3})\delta]$

We just have  $\psi_4 = \psi_5 = 1$  then  $\Delta u = 0$  and we set  $W = 0$ .

*IV.b*  $[\rho + (4 + \frac{5}{3})\delta, \rho + 6\delta]$

We have  $u = \psi_4 u_4 + u_5$ , then  $|u| \geq |u_5| - |u_4|$ .

$$\left| \frac{u_5}{u_4} \right| = \frac{r^k}{(\rho + 5\delta)^k} \geq \left( \frac{\rho + (4 + \frac{5}{3})\delta}{\rho + 5\delta} \right)^k \geq e^{k \ln(1 + \frac{2\delta}{3(\rho + 5\delta)})}$$

Let  $C_1$  such that  $\ln(1 + x) \geq C_1 x$  on  $[0, \frac{2}{15}]$ .

Then we get

$$\left| \frac{u_5}{u_4} \right| \geq e^{C_1 \frac{2k\delta}{3(\rho + 5\delta)}} \geq 1 + C_1 \frac{2k\delta}{3(\rho + 5\delta)},$$

since  $|u| \geq |u_5| - |u_4|$  this gives

$$|u| \geq C_1 \frac{2k\delta}{3(\rho + 5\delta)} |u_4|$$

and then

$$|u_4| \leq \frac{C}{k\delta} |u|. \quad (3.33)$$

We also have

$$\Delta u = 2 \frac{d\psi_1}{dr} \frac{\partial u_4}{\partial r} + \left( \frac{\partial^2 \psi_4}{\partial r^2} + \frac{1}{r} \frac{\psi_4}{\partial r} \right) u_4.$$

Since  $|\frac{\partial u_4}{\partial r}| \leq Cn|u_4|$ , we get

$$|\Delta u| \leq C\frac{n}{\delta}|u_4| + \frac{C}{\delta^2}|u_4| + \frac{C}{\delta}|u_4|.$$

Here again we have obtain from (3.33)

$$|\Delta u| \leq CN^{\frac{3}{2}}|u|.$$

*Step IV.c*  $[\rho + 4\delta, \rho + (4 + \frac{1}{3})\delta]$

This step is similar to step IV.b and therefore the estimation is omitted.

Then, defining  $a_{j+1} = c$  with  $c$  from (3.31) and (3.32), recall that  $n_{j+1} = n_j + k_j = j^2 + 2j + 1$ , and define  $u(r, \varphi) = u_j(r, \varphi)$  for  $\rho_j \leq r \leq \rho_{j+1}$  we have construct on the set

$$I_N = \bigcup_{10 \leq j \leq \sqrt{N}} A_j = \bigcup_{10 \leq j \leq \sqrt{N}} [\rho + 6(j-1)\delta, \rho + 6j\delta] = [\rho + 54\delta, \rho + 6],$$

a solution of  $\Delta u + Wu = 0$  with  $|W| \leq CN^{\frac{3}{2}}$ . Now we extend our construction to the whole  $\mathbb{R}^2$  in the following way: On  $[\rho + 6\delta, +\infty[$  we just keep the harmonic function  $u$  obtained on the last annulus  $A_{\sqrt{N}}$ :  $u = a_N r^{-N} e^{-iN\varphi}$ . On  $[0, \rho + 54\delta]$  we define  $g(r)$  a smooth radial function such that

- $g(r) = r^{n_1}$  in  $[0, \frac{\rho}{4}]$
- $g(r) = r^{-n_1}$  in  $[\frac{3}{4}\rho, \rho + 54\delta]$
- $|g^{(p)}(r)| \leq C\rho^{-1}$ , for all  $r$  in  $[0, \rho + 54\delta]$

The constant  $C$  is the last point doesn't depend on  $N$  ( the gap from  $n_1$  to  $-n_1$  is fixed since  $n_1 = 10$ ). Now define  $u = g(r)e^{-in_1\varphi}$ . On the compact set  $[0, \rho + 10\delta]$  on easily get  $|\frac{\Delta u}{u}| \leq C$  with  $C$  independant on  $N$ . Finally we have constructed on  $\mathbb{R}^2$  a solution of  $\Delta u + Wu = 0$  with  $\|W\|_{L^\infty} \leq CN^{\frac{3}{2}}$  and  $|u(x)| = |x|^{-N}$  for  $|x| \geq \rho + 6$ .

Then we consider the inverse of the stereographic projection :

$$\begin{aligned} \pi : \mathbb{R}^2 &\rightarrow \mathbb{S}^2 \setminus N \\ (x, y) &\mapsto \frac{1}{x^2 + y^2 + 1} (2x, 2y, x^2 + y^2 - 1) \end{aligned}$$

In the Chart  $(\mathbb{S}^2 \setminus (0, 0, 1), \pi^{-1})$ , the canonical metric is written

$$g_{\mathbb{S}^2} = \begin{pmatrix} \frac{4}{(x^2 + y^2 + 1)^2} & 0 \\ 0 & \frac{4}{(x^2 + y^2 + 1)^2} \end{pmatrix}$$

On  $\mathbb{S}^2 \setminus N$ , we have

$$\Delta_{\mathbb{S}^2} = \frac{1}{4}(x^2 + y^2 + 1)^2 \Delta_{\mathbb{R}^2}$$

Let  $\bar{W}$  and  $\bar{u}$  two real valued functions defined on  $\mathbb{S}^2 \setminus (0, 0, 1)$  and  $C$  a positive constant. We consider  $u = \bar{u} \circ \pi^{-1}$  and  $W = \bar{W} \circ \pi^{-1}$ , So we have

$$\left\{ \begin{array}{l} \Delta_{\mathbb{S}^2} \bar{u} = \bar{W} \bar{u} \\ |\bar{W}(x)| \leq C \end{array} \right\} \iff \left\{ \begin{array}{l} \Delta_{\mathbb{R}^2} u = Wu \\ |W(x, y)| \leq \frac{C}{(1 + x^2 + y^2)^2} \end{array} \right.$$



Since the function  $W$  we have constructed is compactly supported with  $|W(x)| \leq CN^{\frac{3}{2}}$ , there exists  $C'$  such that

$$\frac{4}{(1+x^2+y^2)}|W(x,y)| \leq C' N^{\frac{3}{2}}, \quad \forall (x,y) \in \mathbb{R}^2.$$

It follows that the function  $\bar{u}$  is a solution of  $\Delta_{\mathbb{S}^2} \bar{u} = \bar{W} \bar{u}$  on  $\mathbb{S}^2$ , with  $\bar{u}$  vanishing at order  $N$  at the north pole and with  $N \geq \|\bar{W}\|_{\infty}^{2/3}$ .  $\square$

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